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A Fokker–Planck description of *K*-distributed noise

R J A Tough

RSRE, St Andrews Road, Great Malvern, Worcestershire, UK

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Abstract. A Fokker-Planck description of the time evolution of a *K*-distributed noise process is presented, discussed and taken as the basis of an approximate analysis of the correlation properties of the noise. An equivalent set of stochastic differential equations is identified and exploited in a numerical simulation of the noise process.

1. Introduction

In the past decade considerable progress has been made in the characterisation and interpretation of non-Gaussian noise processes in terms of the *K*-distributed process introduced by Jakeman and Pusey [1-3]. (For an up-to-date review see [4].) Much of the earlier theoretical work concentrated on the static statistical properties of the *K*-distributed process, its infinite divisibility and the limiting processes by which it could be derived from random walk models [1, 2]. More recently Jakeman [3] has proposed a model in which coherent scattering from a population of objects subject to processes of birth, death and migration (described formally by a coupled set of rate equations) is shown, in the limit of a large mean number ($\bar{N}$) of scatterers, to yield a *K*-distributed intensity of scattered light. The analysis of the temporal correlation properties inherent in this model is rather delicate, with the limiting $\bar{N} \to \infty$ behaviour having to be extracted from the results of involved calculations. Furthermore Jakeman's model does little to suggest a way in which *K*-distributed noise might be simulated and studied numerically on a computer. In this paper an alternative, but essentially equivalent, formulation of Jakeman's model is developed in which the large $\bar{N}$ limit is taken from the outset in a continuous Fokker-Planck (FP) description of the processes involved. Many techniques for the analysis of FP equations have been developed in recent years [5-7] and so can now be applied to the analysis of a correlated *K*-distributed noise process. Here we apply the Mori projection analysis [8, 9] and the method of adiabatic elimination [5] to investigate the temporal behaviour of the intensity correlation function implicit in our model and regain the generalised Siegert result obtained by Jakeman [3]. Furthermore the stochastic equivalence of an FP equation to a set of stochastic differential equations [5] can be exploited so that a possible method for the numerical simulation of correlated *K*-distributed noise can be identified. However, before introducing the FP description of a *K*-distributed process we review the corresponding descriptions of the birth-death-migration process and the coherent scattering process. This will illustrate how the FP equation provides a natural expression of the $\bar{N} \to \infty$ limit of the rate equation formulation and how a description which makes reference only to the intensity of the scattered radiation is effectively able to encode the phase information implicit in Jakeman's use of a complex electric field in his treatment of the scattering process.

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2. Birth, death and migration: negative binomial and gamma distributed processes

The competing processes of birth, death and migration within a population can, under suitable conditions, establish a negative binomial probability distribution for that population [10]. Fluctuations in this population are non-Gaussian in the sense that their variance is proportional to the square of the mean population $\bar{N}^2$ (rather than linearly dependent upon $\bar{N}$, as for a Gaussian distribution). On adopting the notation of Jakeman's discussion [3], we see that $P_N(\tau)$, the probability of the population consisting of N members at time τ, satisfies the rate equation

$$\frac{\mathrm{d}P_N}{\mathrm{d}\tau}(\tau) = \mu(N+1)P_{N+1}(\tau) - [(\lambda+\mu)N+\nu]P_N(\tau) + [\lambda(N-1)+\nu]P_{N-1}(\tau) \tag{1}$$

where λ, μ and ν characterise the uncorrelated processes of birth, death and migration respectively. For $\mu > \lambda$, (1) has the stationary solution

$$P_N(\infty) = \binom{N+\alpha-1}{N} \frac{(\bar{N}/\alpha)^N}{(1+\bar{N}/\alpha)^{N+\alpha}} \tag{2}$$

where $\binom{a}{b}$ is a binomial coefficient,

$$\alpha = \nu/\lambda \tag{3}$$

and

$$\bar{N} = \frac{\nu}{\mu-\lambda}. \tag{4}$$

The non-Gaussian fluctuation property is manifest in the result

$$\frac{\overline{N^2}}{\bar{N}^2} - 1 = \frac{1}{\alpha} + \frac{1}{\bar{N}}. \tag{5}$$

We now consider the description of the process in the large $\bar{N}$ limit. To do this we introduce the variable x through

$$N = \bar{N}x.$$

As $\bar{N}$ becomes large x becomes an essentially continuous variable. In the same spirit we define

$$\mathscr{P}(x,\tau) \equiv P_N(\tau) = P_{\bar{N}x}(\tau)$$

so that (1) becomes

$$\frac{\partial}{\partial\tau}\mathscr{P}(x,\tau) = \mu\left(x+\frac{1}{\bar{N}}\right)\mathscr{P}\left(x+\frac{1}{\bar{N}},\tau\right) - \left((\lambda+\mu)x+\frac{\nu}{\bar{N}}\right)\mathscr{P}(x,\tau) + \left(\lambda x+\frac{\nu-\lambda}{\bar{N}}\right)\mathscr{P}\left(x-\frac{1}{\bar{N}},\tau\right). \tag{6}$$

An expansion in inverse powers of $\bar{N}$ then gives

$$\frac{1}{\bar{N}}\frac{\partial\mathscr{P}}{\partial\tau}(x,\tau) = \frac{1}{\bar{N}}(\mu-\lambda)\frac{\partial}{\partial x}(x\mathscr{P}(x,\tau)) + \frac{1}{\bar{N}^2}\left(\frac{1}{2}(\mu+\lambda)x\frac{\partial^2}{\partial x^2}\mathscr{P}(x,\tau) + (\mu-\nu+\lambda)\frac{\partial}{\partial x}\mathscr{P}(x,\tau)\right) + \mathrm{o}\left(\frac{1}{\bar{N}^2}\right). \tag{7}$$

From (4) we see that all terms shown explicitly on the right-hand side of (7) are $O(\bar{N}^{-2})$. Finally we scale time by $\bar{N}(t=\tau/\bar{N})$ and note that $\mu \to \lambda$ as $\bar{N} \to \infty$; from (7) it now follows that $\mathscr{P}$ satisfies

$$\frac{\partial \mathscr{P}}{\partial t} = \lambda \frac{\partial^2}{\partial x^2}(x\mathscr{P}) + \nu \frac{\partial}{\partial x}((x-1)\mathscr{P}) \tag{8}$$

which has the form of an archetypal FP equation [5]. It is readily verified that the static solution of (8) is the gamma distribution

$$\mathscr{P}_\infty(x) = \frac{\alpha^\alpha}{\Gamma(\alpha)} \exp(-\alpha x) x^{\alpha-1} \tag{9}$$

normalised in terms of the gamma function Γ.

The derivation of (8) from (1) is facilitated by the scaling property (5) of the fluctuations and can be compared with the following situation discussed by van Kampen [11], Kubo *et al* [12] and others. When making an expansion of a rate or master equation in the reciprocal of some large parameter Ω (the analogue of our $\bar{N}$, and frequently identified with the system size) one decomposes scaled variables (our x) into a 'macroscopic' part which scales as Ω and a term representing fluctuations which, on physical grounds, is taken to scale as $\Omega^{1/2}$. The systematic expansion in terms of inverse powers of Ω is then a matter of some delicacy. In our case the fluctuations scale linearly with the large parameter $\bar{N}$ and are non-Gaussian even in the limit $\bar{N} \to \infty$; the fluctuations in van Kampen's master equation expansion become Gaussian in the limit of large Ω.

Consequently the expansion in $\bar{N}^{-1}$ is straightforward and gives the FP equation directly, rather than through a Mandel transform of the rate equation (1) [13]. We should also note (see van Kampen [7] and Kubo *et al* [12]) that the systematic expansion of a master equation (ultimately to yield an equation of FP type) and the derivations of central limit theorem results in probability theory are intimately connected; by making our expansion in $\bar{N}^{-1}$ and deriving a FP equation we are essentially performing the limiting procedure utilised by Jakeman [3] at the end of his calculations. Thus a FP formulation incorporates the limiting behaviour from the outset in a continuous (rather than discrete) model.

3. Coherent scattering—the Rayleigh distribution

In the standard discussion of coherent scattering (see, for example, Pusey [14]) the scattered electric field vector ε is represented as a sum of randomly phased terms

$$\varepsilon(t) = \sum_{j=1}^{N} \exp(i\phi_j(t)) \tag{10}$$

where, for convenience, we have assumed a constant and normalised scattering power. The statistical properties of the scattered radiation are implicit in those of the phases $\phi_j(t)$ and in the identification of the intensity $I(t)$ through

$$I(t) = \varepsilon(t)\varepsilon(t)^*. \tag{11}$$

Following and extending the discussion of Pusey [14] we form

$$C_{n,m}(t) = \langle I^n(t) I^m(0) \rangle / \langle I \rangle^{n+m} \tag{12}$$

in the large N limit, in which overcounting corrections tend to zero in the normalised correlation function (12). From (10) and (11) we see that

$$C_{n,m}(t) = N^{-(n+m)} \sum_{\substack{j_1 \ldots j_n \\ k_1 \ldots k_n \\ l_1 \ldots l_m \\ q_1 \ldots q_m}} \langle \exp[\mathrm{i}(\phi_{j_1}(t) - \phi_{k_1}(t) + \phi_{j_2}(t) - \phi_{k_2}(t) + \ldots + \phi_{j_n}(t) - \phi_{k_n}(t)$$

$$+ \phi_{l_1}(0) - \phi_{q_1}(0) + \ldots + \phi_{l_m}(0) - \phi_{q_m}(0))] \rangle \tag{13}$$

with the phase factors paired up as shown with n pairs formed from values at time t and m pairs with zero time values; there are $n! \times m!$ ways of forming these pairs. It is assumed that the phase terms are statistically independent so that

$$\begin{aligned} &\langle \exp \mathrm{i}(\phi_k(t) - \phi_l(t')) \rangle = g_1(t - t')\delta_{k,l} \\ &g_1(0) = 1. \end{aligned} \tag{14}$$

Thus terms in which all $j_\mu = k_u$ and $l_u = q_u$ will give a unit contribution. If $j_u \neq k_u$, $l_v \neq q_v$ for r of the $n(m)$ values of $u(v)$ there will be a contribution of $|g_1(t)|^{2r}$ to the normalised correlation function. The combinatorial factor corresponding to the number of ways of picking such pairs is

$$\frac{n!}{(n-r)!\,r!} \frac{m!}{(m-r)!\,r!}$$

so that the normalised correlation function $C_{n,m}(t)$ is given by

$$C_{n,m}(t) = n!\,m! \sum_{r=0}^{\min(n,m)} \frac{n!\,m!}{(n-r)!(m-r)!(r!)^2} |g_1(t)|^{2r} \tag{15}$$

as $N \to \infty$. This provides us with a generalisation of the Siegert relation [14] to the more general correlation function $C_{n,m}(t)$.

As is well known [14] the pairing argument given above, specialised to zero time, gives the following values for the normalised moments of the intensity distribution:

$$\langle I^n \rangle / \langle I \rangle^n = n!. \tag{16}$$

(This can also be shown directly from (15) by setting $|g_1(0)|^2 = 1$, identifying $C_{n,m}(0)$ as $n!\,m!\,{}_2F_1(-n, -m; 1; 1)$ and employing Gauss' theorem to evaluate the hypergeometric function ${}_2F_1$ as $(n+m)!/(n!\,m!)$ [15].) The moments (16) are consistent with the intensity having a Rayleigh distribution

$$P(I) = \frac{1}{\langle I \rangle} \exp(-I/\langle I \rangle). \tag{17}$$

We will now show that a FP description of the intensity statistics is possible which in effect encodes the statistical properties of the phase used in the derivation of (15), subject to the assumption that $|g_1(t)|^2$ decays as a single exponential. We see from (8) and (9) that the Rayleigh distribution (17) is a stationary solution P_∞ of the FP equation

$$\frac{\partial^2}{\partial z^2}(zP) + \frac{\partial}{\partial z}[(z-1)P] = \frac{\partial P}{\partial t} \tag{18}$$

where $z = I/\langle I \rangle$. Correlation functions such as $C_{n,m}(t)$ of (13) can be written in terms of $G(z, t \mid z_0)$, the propagator or fundamental solution of (18), which can be interpreted

as the conditional probability that, given a value z_0 at time zero, a random variable takes a value z at time t. $G(z, t|z_0)$ satisfies (18) and the initial condition

$$G(z, 0|z_0) = \delta(z - z_0). \tag{19}$$

Given $G(z, t|z_0)$ we can then write

$$\langle z^n(t) z^m(0)\rangle = \int dz \int dz_0\, z^n z_0^m G(z, t|z_0) P_\infty(z_0). \tag{20}$$

The expansion of $G(z, t|z_0)$ in terms of the eigenfunctions of the adjoint

$$\tilde{\mathscr{L}} = z\frac{d^2}{dz^2} + (1-z)\frac{d}{dz}$$

of the FP operator defined in (18) is a standard procedure (see, for example, the paper of Wong [16])

$$G(z, t|z_0) = e^{-z} \sum_{r=0}^{\infty} L_r(z) L_r(z_0) \theta(t)^r. \tag{21}$$

Here $\theta(t) = \exp(-t)$ and L_r is the Laguerre polynomial defined by

$$L_r(z) = \frac{e^z}{r!}\frac{d^r}{dz^r}(e^{-z} z^r).$$

From (20) we find that

$$\langle z(t)^n z(0)^m\rangle = \sum_r \theta(t)^r \int_0^\infty dz\, z^n\, e^{-z} L_r(z) \int_0^\infty dz_0\, z_0^m\, e^{-z_0} L_r(z_0)$$

the integrations over z, z_0 are standard

$$\int_0^\infty dz\, z^n\, e^{-z} L_r(z) = (-1)^r \frac{(n!)^2}{(n-r)!\,r!} \qquad n \geqslant r$$

$$= 0 \qquad n < r \tag{22}$$

and yield

$$\langle z(t)^n z(0)^m\rangle = n!\,m! \sum_{r=0}^{\min(n,m)} \frac{n!\,m!}{(n-r)!\,(m-r)!\,(r!)^2}\,\theta(t)^r. \tag{23}$$

This result coincides with (15) if the exponentially decaying $\theta(t)$ is identified with the square modulus of the field correlation function $|g_1(t)|^2$, and we recognise that we work in the limit of a large number of scatterers (cf the FP treatment of the gamma process). We see that the orthogonal polynomials $L_r(z)$, and ultimately the structure of the FP operator in (18), encode, through the expansion (21) of the propagator $G(z, t|z_0)$ and the moment result (22), the combinatorial factors implicit in the assumed statistical properties (14) of the phases $\phi_j(t)$ in (10). An exponential decay in $|g_1(t)|^2$ is consistent (through Doob's theorem [17]) with ε being a complex Gauss-Markov process: the Markov property is also implicit in the FP formulation of a process [5]. Should $|g_1(t)|^2$ not have this simple exponential time dependence (with ε no longer being a Markov process), the combinatorial arguments leading to (15) carry through and an expansion analogous to (21), with $\theta(t)$ replaced by $|g_1(t)|^2$, will still hold.

However such a conditional probability, which can be expressed more compactly through the use of Mehler's summation as

$$\frac{1}{1-|g_1(t)|^2}\exp\left(-\frac{(z+|g_1(t)|^2 z_0)}{1-|g_1(t)|^2}\right)I_0\left(\frac{2|g_1(t)|(zz_0)^{1/2}}{1-|g_1(t)|^2}\right)$$

where I_ν is a modified Bessel function, can no longer determine the properties of many time correlation functions such as $\langle z(t_1)z(t_2)z(t_3)\rangle$ as $G(z, t\,|\,z_0)$ does for a Markov process. As the process is no longer Markov a simple FP description would also not be appropriate.

4. *K*-distributed noise—a Fokker–Planck description

We have now seen how the principal ingredients of a K-distributed noise model—the large mean population limit of a birth, death and migration process and coherent Rayleigh scattering—can each be accommodated in a FP description. It remains to fuse the two together. In doing this we shall be guided at the outset by a formal intuition; the close connection between our model and that of Jakeman [3] should become apparent later.

Consider a Rayleigh distributed process z, with a current mean value x, whose distribution function is (cf (17)) $\exp(-z/x)/x$. We now take x to be a gamma-distributed variable, so that the stationary joint distribution of x, z may be written as

$$P_\infty(x, z) = \frac{1}{\Gamma(\nu+1)}x^{\nu-1}\exp(-x)\exp(-z/x). \tag{24}$$

The marginal distribution of z is now a K distribution:

$$\int_0^\infty dx\, x^{\nu-1}\exp(-x)\exp(-z/x) = 2z^{\nu/2}K_\nu(2\sqrt{z}) \tag{25}$$

(K_ν is a modified Bessel function of the second kind). It is this integral identity which lies at the heart of Jakeman's model; the implicit factorisation of a K-distributed process into a Rayleigh process with a gamma-distributed mean also underpins the use of the K model in the analysis of land [18] and sea [19] clutter. Thus to provide a FP description of the K-distributed process z we seek an equation in the variables x and z which is of FP form and has $P_\infty(x, z)$ of (24) as its stationary solution. We also require that there be as strong a resemblance as possible between this FP equation and those describing the gamma- and Rayleigh-distributed processes ((8) and (18) respectively) and that significant contact can be made between our description and Jakeman's model. A FP equation satisfying these criteria is

$$\frac{\partial}{\partial t}P(x, z; t) = \underbrace{\mathscr{A}\left(\frac{\partial^2}{\partial x^2}(xP)+\frac{\partial}{\partial x}[(x-\nu-z/x)P]\right)}_{\text{(i)}}$$

$$+\underbrace{\mathscr{B}\left(\frac{\partial^2}{\partial z^2}(zP)+\frac{\partial}{\partial z}[(z/x-1)P]\right)}_{\text{(ii)}} = \mathscr{L}P \tag{26}$$

which we see does indeed have (24) as a stationary solution. This feature is in itself worthy of comment as it is not generally the case for a FP equation describing the

coupled behaviour of two or more random variables to have a stationary solution which is accessible to direct quadrature [5]. The existence of such a solution allows the FP operator $\mathscr{L}$ to be cast into self-adjoint form through a transformation of the type

$$\mathscr{H} = P_\infty^{-1/2} \mathscr{L} P_\infty^{1/2}$$

and so ensures the existence of a complete orthonormalised set of eigenfunctions of that operator [5]. It is therefore possible, in principle, to expand the fundamental solution of (26) in terms of this complete set, just as was done, for example, in (21). However, an extension of the analysis of Wong and Thomas [20] indicates that, in contrast to the cases considered in the foregoing two sections, it is not possible to obtain eigenfunctions of the FP operator in (26) in the form of orthogonal polynomials in the variables x and z. Indeed, very little is known about the eigenfunctions and the eigenvalue spectrum of two-dimensional operators of this type. Nonetheless, given the increasing importance of the K-distributed noise model, analysis of these difficult problems should prove worthwhile. Furthermore, as we shall see in the next two sections, it is possible to investigate the properties of the FP model by using approximate methods and by numerical simulation.

Before discussing these implications of (26) further we consider a possible physical interpretation of the variables z and x, noting that our proposed FP equation is made up of terms reminiscent of those occurring in the description of the gamma and Rayleigh processes. Thus the term (ii) strongly resembles (18) while (i) is, apart from the non-linear coupling in z/x, the FP operator appropriate to the description of a gamma process. These observations suggest that z be identified with the intensity of light scattered coherently from an illuminated area, containing correlated scatterers whose scattering power or cross section is given by x. The factors $\mathscr{A}$ and $\mathscr{B}$ incorporate, through their reciprocals, the differing rates of decorrelation of the scatterers and the scattered light into the model. The non-linear coupling between the z and x variables through the term z/x is necessary to maintain (24) as the stationary solution of our FP equation. The physical interpretation of this coupling in (ii) as a varying mean intensity of the scattering process z is straightforward; the interpretation of the corresponding non-linear term in (i) is not as evident as the formal requirement that it be present to ensure the correct stationary solution. As we shall see, in the limit that z decorrelates much more rapidly than x (as is the case in the physical realisation of Jakeman's model) the two processes effectively decouple and the physical interpretation of x and z is obvious. The marginal distribution of x derived from (24) is the gamma distribution

$$\int_0^\infty \mathrm{d}z \exp(-x) x^{\nu-1} \exp(-z/x) = x^\nu \exp(-x) \tag{27}$$

identical with that obtained by Jakeman for the intensity resulting from the incoherent illumination of a negative binomial distributed population in the limit of large N. Contributions to an incoherently scattered intensity do more than count the scatterers in the illuminated area so that the total intensity can be identified with N of equation (1). This again bears out our interpretation of x as a normalised cross section of scatterers in the illuminated area. Thus we have been able to establish, albeit only by plausibility arguments, a reasonable connection between the variables z and x of (26) and the coherently scattered intensity and the cross section of scatterers in an illuminated area.

5. Correlation in a *K*-distributed process—an approximate analysis

Having postulated a FP description of a *K*-distributed random process we will now analyse the autocorrelation properties of this process, using a standard technique from statistical mechanics (the Mori projection operator method [8, 9]) adapted to FP dynamics of Ackerson [21]. The result we obtain has the same form as, and is essentially identical with, the generalised Siegert relation derived by Jakeman [3].

The Mori projection operator method enables us to generate equations of motion for correlation functions of a set of variables, by projecting the equations of motion of these variables onto their values at some (earlier) time. In our case we chose the variables

$$\delta z = z - \int_0^\infty \mathrm{d}x\, \mathrm{d}z\, z P_\infty(x, z) = z - (\nu + 1)$$

$$\delta x = x - \int_0^\infty \mathrm{d}x\, \mathrm{d}z\, x P_\infty(x, z) = x - (\nu + 1).$$

The FP equation (26) describes the temporal evolution of the propagator $G(x, z, t \,|\, x_0, z_0)$ through

$$\frac{\partial G}{\partial t} = \mathscr{L} G \tag{28}$$

subject to the initial condition $G(x, z, 0 \,|\, x_0, z_0) = \delta(x - x_0)\delta(z - z_0)$. We wish to derive equations of motion for the matrix of correlation functions formed from the vector $\boldsymbol{\phi}^{\mathrm{T}} = (\delta x, \delta z)$:

$$\langle \boldsymbol{\phi}(0)\boldsymbol{\phi}(t)^{\mathrm{T}} \rangle = \mathbb{C}(t) = \begin{pmatrix} \langle \delta x(0)\delta x(t) \rangle & \langle \delta z(0)\delta x(t) \rangle \\ \langle \delta x(0)\delta z(t) \rangle & \langle \delta z(0)\delta z(t) \rangle \end{pmatrix}. \tag{29}$$

Here, and subsequently, the angular brackets $\langle\ \rangle$ denote an average over the stationary distribution (24). Each element of the matrix $\mathbb{C}(t)$ may be written in terms of $G(x, z, t \,|\, x_0, z_0)$ (cf (20)); when we write the formal solution of (28) as

$$G(x, z, t \,|\, x_0, z_0) = \exp(\mathscr{L}t)\delta(x - x_0)\delta(z - z_0)$$

we find, for a typical element

$$\langle \delta z(0)\delta z(t) \rangle = \int\int \mathrm{d}x\, \mathrm{d}x_0\, \mathrm{d}z\, \mathrm{d}z_0\, \delta z \delta z_0 (\exp(\mathscr{L}t)\delta(x - x_0)\delta(z - z_0)) P_\infty(x_0, z_0)$$

which, after successive partial integrations, can be written as

$$\begin{aligned} \langle \delta z(0)\delta z(t) \rangle &= \int \mathrm{d}x\, \mathrm{d}z\, P_\infty(x, z)(\delta z \exp(\tilde{\mathscr{L}}t)\delta z) \\ &= \langle \delta z \exp(\tilde{\mathscr{L}}t)\delta z \rangle \end{aligned} \tag{30}$$

where

$$\tilde{\mathscr{L}} = \mathscr{A}\left(x\frac{\partial^2}{\partial x^2} + (\nu - x + z/x)\frac{\partial}{\partial x} \right) + \mathscr{B}\left(z\frac{\partial^2}{\partial z^2} + (1 - z/x)\frac{\partial}{\partial z} \right) \tag{31}$$

is the adjoint of the FP operator in (26). Thus an equation of motion for $\langle \delta z(0)\delta z(t)\rangle$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \delta z \exp(\tilde{\mathscr{L}}t)\delta z\rangle = \langle \delta z \exp(\tilde{\mathscr{L}}t)\tilde{\mathscr{L}}\delta z\rangle \tag{32}$$

with similar results holding for the other elements of $\mathbb{C}(t)$. Following the standard Mori strategy [8, 9] we define an operator $\mathbb{P}$ which projects onto the variables δx and δz

$$\mathbb{P}A = \langle A\boldsymbol{\phi}^{\mathrm{T}}\rangle \cdot \langle \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\rangle^{-1} \cdot \boldsymbol{\phi}^{\mathrm{T}}. \tag{33}$$

Here $\langle \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\rangle$ is the zero time value of $\mathbb{C}(t)$:

$$\mathbb{C}(0) = \begin{pmatrix} (\nu+1) & (\nu+1) \\ (\nu+1) & (\nu+1)(\nu+3) \end{pmatrix} \tag{34}$$

whose inverse

$$\langle \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\rangle^{-1} = \begin{pmatrix} \dfrac{\nu+3}{(\nu+2)(\nu+1)} & \dfrac{-1}{(\nu+1)(\nu+2)} \\ \dfrac{-1}{(\nu+1)(\nu+2)} & \dfrac{1}{(\nu+1)(\nu+2)} \end{pmatrix}. \tag{35}$$

$\mathbb{Q}$ is the complement of $\mathbb{P}$, i.e.

$$\mathbb{P}+\mathbb{Q} = \mathbb{1}$$

the identity operator. We now write

$$\exp(\tilde{\mathscr{L}}t)\tilde{\mathscr{L}} = \exp(\tilde{\mathscr{L}}t)(\mathbb{P}+\mathbb{Q})\tilde{\mathscr{L}} \tag{36}$$

and decompose $\exp(\tilde{\mathscr{L}}t)$ as follows:

$$\exp(\tilde{\mathscr{L}}t) = \exp(\mathbb{Q}\tilde{\mathscr{L}}t) + \int_0^t \mathrm{d}t' \exp[\tilde{\mathscr{L}}(t-t')]\mathbb{P}\tilde{\mathscr{L}} \exp(\mathbb{Q}\tilde{\mathscr{L}}t'). \tag{37}$$

On incorporating these results into (32) we obtain the following equation of motion for $\mathbb{C}(t)$:

$$\frac{\mathrm{d}\mathbb{C}}{\mathrm{d}t} = \mathbb{C}(t)\langle \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\rangle^{-1}\langle \boldsymbol{\phi}\tilde{\mathscr{L}}\boldsymbol{\phi}^{\mathrm{T}}\rangle + \int_0^t \mathrm{d}t_1\, \mathbb{C}(t-t_1)\langle \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\rangle^{-1}\langle \mathbb{Q}\tilde{\mathscr{L}}\boldsymbol{\phi}(0) \exp(\mathbb{Q}\tilde{\mathscr{L}}t_1)\mathbb{Q}\tilde{\mathscr{L}}\boldsymbol{\phi}^{\mathrm{T}}(0)\rangle. \tag{38}$$

The manipulations leading to (38) are a formal restructuring of the equation of motion (32). A first approximation to the behaviour of the correlation matrix $\mathbb{C}(t)$ which is exact at short times can be obtained by neglecting the second 'memory' term in (38) which incorporates those effects of non-linearity not represented in the linearised equations making up the first term. Within the context of the diffusive dynamics of suspensions, which are described by FP equations analogous to (26), this 'first cumulant' approximation is widely used [22, 23]. Thus we evaluate $\langle \boldsymbol{\phi}^{\mathrm{T}}\tilde{\mathscr{L}}\boldsymbol{\phi}\rangle$ as

$$\begin{pmatrix} -\mathscr{A}(\nu+1) & 0 \\ 0 & -\mathscr{B}(\nu+1) \end{pmatrix}$$

and make use of (35) to obtain

$$\frac{\mathrm{d}\mathbb{C}}{\mathrm{d}t} = \boldsymbol{\Omega}\mathbb{C} \tag{39}$$

where $\mathbf{\Omega}$ is the frequency matrix

$$\mathbf{\Omega} = \begin{pmatrix} \dfrac{-\mathscr{A}(\nu+3)}{(\nu+2)} & \dfrac{\mathscr{A}}{(\nu+2)} \\ \dfrac{\mathscr{B}}{(\nu+2)} & \dfrac{-\mathscr{B}}{(\nu+2)} \end{pmatrix}. \tag{40}$$

Laplace transformation of (39) and a subsequent inversion yield the transformed correlation matrix $\tilde{\mathbf{C}}(s)$. Our principal interest is in the autocorrelation function of the process δz whose transform can be expressed as

$$\tilde{C}_{zz}(s) = \frac{1}{(s-s_+)(s-s_-)}\left(\mathscr{B}\frac{(\nu+1)}{(\nu+2)} + s(\nu+1)(\nu+3) + \mathscr{A}\frac{(\nu+3)^2}{(\nu+2)}\right) \tag{41}$$

where s_+ and s_- are roots of the equation

$$s^2 + s[\mathscr{B} + \mathscr{A}(\nu+3)]/(\nu+2) + \mathscr{A}\mathscr{B}/(\nu+2) = 0$$

i.e.

$$s_\pm = \frac{1}{2(\nu+2)}\{-[\mathscr{B} + \mathscr{A}(\nu+3)] \pm [[\mathscr{B} + \mathscr{A}(\nu+3)]^2 - 4\mathscr{A}\mathscr{B}(\nu+2)]^{1/2}\}. \tag{42}$$

Equation (41) can now be inverted quite readily to express $C_{zz}(t)$ as a sum of two exponential terms. To make contact with Jakeman's model we note that the timescale for the decay of correlation in the coherent scattering process (characterised by the reciprocal of the multiplicative factor $\mathscr{B}$ in (26)) is much shorter than a typical correlation time for fluctuations in the number of scatterers themselves (here characterised by the reciprocal of $\mathscr{A}$). Thus we should consider the result (41) in the limit $\mathscr{B} \to \infty$ for which s_+ and s_- tend to the values

$$\begin{aligned} s_+ &= -\mathscr{A} \\ s_- &= -\mathscr{B}/(\nu+2) \end{aligned} \tag{43}$$

which characterise a slow and a fast mode respectively. In this limit we find that

$$C_{zz}(t) = (\nu+1)\exp(-\mathscr{A}t) + (\nu+2)(\nu+1)\exp[-\mathscr{B}t/(\nu+2)] \tag{44}$$

from which $C_{zz}(0) = (\nu+1)(\nu+3)$, as we would hope (cf (34)). The first, slowly decaying, term can be identified with the number fluctuation term in Jakeman's generalised Siegert relation, while the second rapidly decaying term can be identified with the speckle or field correlation term. The amplitudes of these two terms given by (44) and Jakeman are identical. This underlines the intimate connection between the two models. An alternative approximate analysis, which yields a result equivalent to (44), can be performed using the method of adiabatic elimination [5]. This is outlined in the appendix.

6. The numerical simulation of *K*-distributed noise

A Fokker-Planck equation such as (23) provides a description of a random process, in terms of the propagator $G(x, z, t \mid x_0, z_0)$, whose probabilistic element is the interpretation of $G(x, z, t \mid x_0, z_0)$ as a conditional probability. An alternative description of such a process is provided by a set of stochastic differential equations (SDE) which

incorporate random noise terms explicitly. The archetypal SDE is the Langevin equation familiar from the description of Brownian motion [17]; a set of SDE sufficiently general for our purposes is

$$\frac{\mathrm{d}}{\mathrm{d}t} x_i = F_i(\boldsymbol{x}) + \sum_j g_{ij}(\boldsymbol{x}) f_j(t). \tag{45}$$

Here F_i and g_{ij} are, in general, non-linear functions of the vector of random variables $\boldsymbol{x}$ while $f_i(t)$ are a set of Gaussian white noise random variables with the temporal correlation property

$$\langle f_i(t) f_j(t') \rangle = 2\delta(t-t')\delta_{i,j}. \tag{46}$$

The g_{ij} model the dependence of the fluctuations in the system represented by (45) on the configuration of that system. In the standard Langevin model of Brownian motion [17], the g_{ij} are simply constants and (45) is then said to represent a system with additive noise. When, in the more general case, g_{ij} depend explicitly on $\boldsymbol{x}$ (45) represents a system with multiplicative noise.

The equivalence of FP and SDE descriptions of a random process has been discussed exhaustively in the literature in several contexts [5, 7]. For systems with additive noise the connection between the two descriptions is unambiguous and is described clearly and concisely in the classic reviews of Chandrasekhar [24] and Wang and Uhlenbeck [17]. Rather more controversy has been attached to the interpretation of SDE with multiplicative noise and to the identification of their stochastically equivalent FP equation. Reference can be made to the review of Lindenberg *et al* [25] and the text of Gardiner [5] for a thorough discussion of this topic. Here we will adopt without further comment the Itô interpretation of the SDE (45) and identify their stochastically equivalent FP equation as

$$\frac{\partial}{\partial t} P(\boldsymbol{x}, t) = \sum_{i,j,k} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (g_{ik}(\boldsymbol{x}) g_{jk}(\boldsymbol{x}) P(\boldsymbol{x}, t)) - \sum_i \frac{\partial}{\partial x_i} (F_i(x) P(\boldsymbol{x}, t)). \tag{47}$$

Consistent adherence to Itô's convention renders any controversy irrelevant. Thus, for example, we see by inspection that the single SDE

$$\mathrm{d}z/\mathrm{d}t = (1-z) + z^{1/2} f(t) \tag{48}$$

is stochastically equivalent to the FP equation (14) describing the Rayleigh-distributed process z. Similarly the two coupled SDE

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= \mathscr{A}(\nu - x + z/x) + \mathscr{A}^{1/2} x^{1/2} f_1(t) \\ \mathrm{d}z/\mathrm{d}t &= \mathscr{B}(1 - z/x) + \mathscr{B}^{1/2} z^{1/2} f_2(t) \end{aligned} \tag{49}$$

are readily shown to be stochastically equivalent to the FP equation (23).

The SDE (49) can, on integration, provide us with a possible method for the numerical simulation of K-distributed noise. Some care is needed in selecting a suitable method for effecting this quadrature. The uncritical application of numerical methods used for the integration of ordinary differential equations in this context can result in processes corresponding to either the Itô or Stratonovich interpretations of the SDE in question, as well as other 'unidentified intermediate solutions' [34]. A simple (but rather slowly convergent) algorithm which explicitly incorporates the Itô interpretation

of (49) is

$$x(t+\Delta t) = x(t) + \mathscr{A}\left(\nu - x(t) + \frac{z(t)}{x(t)}\right)\Delta t + (2\mathscr{A}x(t)\,\Delta t)^{1/2} r_1 + o(\Delta t) \tag{50a}$$

$$z(t+\Delta t) = z(t) + \mathscr{B}\left(1 - \frac{z(t)}{x(t)}\right)\Delta t + (2\mathscr{B}z(t)\,\Delta t)^{1/2} r_2 + o(\Delta t) \tag{50b}$$

where r_1 and r_2 are uncorrelated random numbers drawn from a Gaussian distribution of zero mean and unit variance. Equations (50) can be iterated forward in time on a computer, taking Δt to be as small as possible, consistent with available time and computational power. This will generate trajectories in (x, z) space which will, on average, be distributed in accordance with (24). By sampling z along these trajectories and allowing x to evolve subject to (50) we are sampling the marginal distribution of z, which we know from (25) to be a K distribution. Speaking loosely, we might say that our procedure provides us with a Monte Carlo realisation of this integral representation which, as we have already noted, lies at the heart of Jakeman's model. It is quite pleasing to note that, while K_ν is a 'special' function, the proposed method of simulation does not itself require the evaluation of anything more recondite than a square root.

The analysis of the previous section has linked the factors $\mathscr{A}$ and $\mathscr{B}$ with effective rates of decay of the number fluctuation and speckle contributions to the autocorrelation function of z. In the limit $\mathscr{B} \to \infty$ in which fast fluctuations in z decorrelate in a time over which x varies only imperceptibly, the method of simulation proposed can be simplified considerably. By averaging (50*b*) over the timescale of the rapid fluctuations in z (an average we denote by $\bar{}$) we obtain

$$\overline{\frac{\mathrm{d}z}{\mathrm{d}t}} = 0 = 1 - \left(\frac{\bar{z}}{x}\right)$$

or

$$\left(\frac{\bar{z}}{x}\right) = 1$$

which we may now substitute into (50*a*) to give

$$\mathrm{d}x/\mathrm{d}t = \mathscr{A}(\nu + 1 - x) + \mathscr{A}^{1/2} x^{1/2} f_1(t) \tag{51}$$

(see also the discussion in the appendix).

Equation (51) can be recognised as the SDE generating gamma-distributed noise, which can be implemented on the computer through

$$x(t+\Delta t) = x(t) + \mathscr{A}(\nu + 1 - x(t))\,\Delta t + (2\mathscr{A}x(t)\,\Delta t)^{1/2} r + o(\Delta t). \tag{52}$$

Thus if we generate correlated gamma noise by this method and at each step evaluate z from uncorrelated Rayleigh noise of unit mean (obtained most straightforwardly by forming $\frac{1}{2}(r_1^2 + r_2^2)$, where r_1 and r_2 are defined as in (50)) by multiplying it by the current value of x, we again generate a distribution of x and z of the form (24) from which K-distributed z can be sampled. In this method the time step Δt chosen for the integration need only be small compared with the reciprocal of $\mathscr{A}$; for the more general method Δt has to be small compared with the reciprocal of the larger of $\mathscr{A}$ and $\mathscr{B}$. Conversely the more general method should give more flexibility in modelling the correlation properties of the simulated noise.

The method suggested for the integration of the SDE is undeniably crude, but has been widely used in the investigation of physical systems described by SDE with multiplicative noise [26, 27]. Indeed the numerical integration of such SDE by more sophisticated methods is a virtually unexplored subject, although some progress has been made recently for the special case of additive noise [28, 29]. We note that the values of z and x are confined to the positive quadrant (as would be consistent with their interpretation as a scattered intensity and a scattering power, respectively), the edges of which form a natural boundary across which a point (x, z) cannot pass in the course of a simulation. This property is manifest in the SDE (49). Thus, for the z variable we see that, as long as $x \neq 0$, the value of dz/dt as $z \to 0$ is positive and so z will tend to increase; in particular the amplitude of the random multiplicative noise term will vanish as the natural barrier at $z = 0$ is approached. Similarly dx/dt is positive for $z \neq 0$, $x \to 0$, again ensuring that x remains positive. This apparently straightforward inclusion of natural boundaries into the formulation of the problem is peculiar to systems with multiplicative noise. The incorporation of ('artificial') barriers into the stochastic simulation of systems with additive noise is not possible merely by a modification of the SDE [30] and requires the use of special techniques [31]. We note that in the strongly non-Gaussian limit as $\nu \to -1$ the integration of the SDE (50) is rather delicate and the natural boundary at $x = 0$ becomes unstable with respect to numerical errors. In the long term the resolution of this problem must lie in the development of improved algorithms for the numerical integration of SDE; in the short term we note that the contracted simulation method based on (52) remains stable for $-1 < \nu < 0$ and that the natural barrier at $x = 0$ is reinforced in the set of SDE:

$$\begin{aligned} dx/dt &= \mathscr{A}[\nu + 1 - x(1+y) + z/x] + (\mathscr{A}x)^{1/2} f_1(t) \\ dy/dt &= \mathscr{B}(1 - xy) + (\mathscr{B}y)^{1/2} f_2(t) \\ dz/dt &= \mathscr{C}(1 - z/x) + (\mathscr{C}z)^{1/2} f_3(t). \end{aligned} \tag{53}$$

These SDE are stochastically equivalent to a FP equation whose stationary solution is $x^{\nu} \exp[-x(1+y) - z/x]$, from which we see that z is still K distributed through

$$2z^{\nu/2} K_{\nu}(2\sqrt{z}) = \int_0^{\infty} dx \int_0^{\infty} dy\, x^{\nu} \exp(-x) \exp(-z/x) \exp(-xy). \tag{54}$$

Some properties of numerically simulated correlated K-distributed noise, generated by the integration of Langevin equations and by other methods, are discussed elsewhere [32]. Here we merely present a sample of simulated K-distributed noise (figure 1),

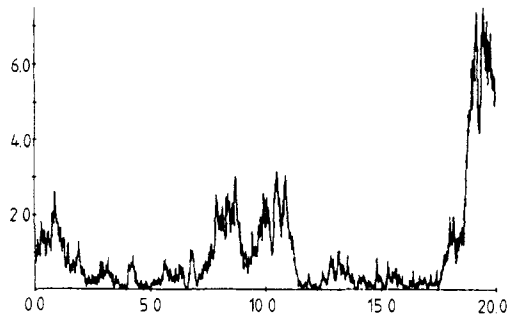

Figure 1. A sample of numerically simulated correlated K-distributed noise ($\nu = -0.4$).

which can be compared with Rayleigh noise (figure 2) and experimentally detected K-distributed noise (figure 3). The last of these was obtained by the illumination by laser light of an electrically excited 'dynamic scattering' layer of nematic liquid crystal [33]. We note the qualitative similarity of figures 1 and 3, in particular the occurrence of periods of relative quiescence and of suddenly occurring bursts, features which are absent from the sample of simulated correlated Rayleigh noise.

7. Conclusions

A Fokker–Planck description of a correlated K-distributed noise process has been set up and discussed. The connection between this formulation and Jakeman's rate-equation model has been demonstrated; in particular a simple projection operator analysis produces a result equivalent to the generalised Siegert relation decomposing the intensity correlation function into a rapidly varying speckle and a slower number fluctuation term. Stochastic differential equations equivalent to the Fokker–Planck equation have been presented and can be used as the basis of a numerical simulation of correlated K-distributed noise. It is hoped that, having cast the description of a K-distributed noise process into a standard Fokker–Planck form and identified a method for its numerical simulation, it will now be possible to analyse this practically important non-Gaussian noise process in greater detail.

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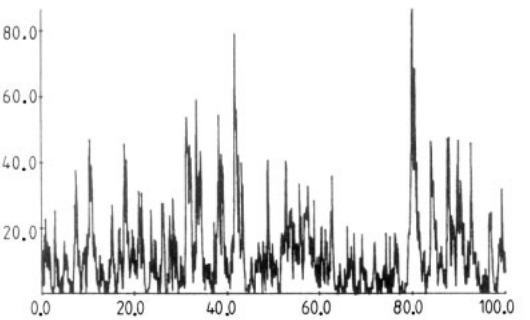

Figure 2. A sample of numerically simulated Rayleigh noise.

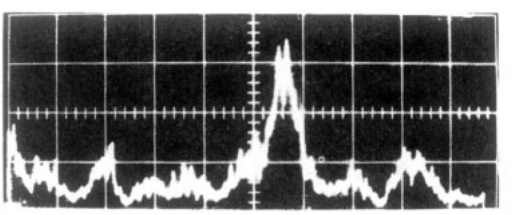

Figure 3. Intensity of light scattered by a turbulent liquid crystal layer whose statistics are fitted by a K distribution with $\nu = -0.4$.

Appendix

The result (44) and the method of simulation (52) are appropriate to the limit in which the variable z decorrelates on a timescale over which x changes imperceptibly. In this appendix we apply the method of adiabatic elimination to investigate this limit and to substantiate the intuitive derivation of (51). Our treatment closely follows that of Gardiner (reference [5], ch 6), to which reference can be made for further details of the method.

We consider the FP equation (26), which we can rewrite as

$$(\partial P/\partial t)(x, z, t) = (\mathscr{B}L_1 + L_2 + L_3)P \tag{A1}$$

having set $\mathscr{A} = 1$. Here

$$L_1 = \frac{\partial^2}{\partial z^2}(z\ ) + \frac{\partial}{\partial z}\Big((z/x - 1)\Big) \tag{A2}$$

$$L_2 = \frac{\partial}{\partial x}\Big((1 - z/x)\Big) \tag{A3}$$

$$L_3 = \frac{\partial^2}{\partial x^2}(x\ ) + \frac{\partial}{\partial x}\Big((x - \nu - 1)\Big) \tag{A4}$$

and derive an equation for

$$\pi(x, t) = \int \mathrm{d}z\, P(x, z, t) \tag{A5}$$

in the limit where $\mathscr{B} \to \infty$.

We identify a solution $\bar{P}_x(z)$ of

$$L_1 \bar{P}_x(z) = 0$$

as

$$\bar{P}_x(z) = (1/x)\exp(-z/x) \tag{A6}$$

and define a projection operator through

$$\mathbb{P}f(x, z) = \bar{P}_x(z) \int \mathrm{d}z\, f(x, z). \tag{A7}$$

In particular we note that

$$\mathbb{P}P(x, z, t) = \bar{P}_x(z)\pi(x, t).$$

It is straightforward to show that

$$\mathbb{P}L_1 = L_1\mathbb{P} = 0 \tag{A8}$$

and, as

$$\int \mathrm{d}z\, z\bar{P}_x(z) = x$$

that

$$\mathbb{P}L_2\mathbb{P} = 0. \tag{A9}$$

The results (A8) and (A9) are those essential for the implementation of the adiabatic elimination procedure described by Gardiner. By defining

$$v(x, t) = \bar{P}_2(z)\pi(x, t) = \mathbb{P}P(x, z, t) \tag{A10}$$

$$w(x, t) = (1-\mathbb{P})P(x, z, t) \tag{A11}$$

deriving their coupled equations of motion from (A1) with (A9) and (A10), Laplace transformation and the subsequent elimination of w we find that (cf Gardiner [5], e.g. (6.6.83))

$$s\tilde{v}(s) = \{L_3 - \mathscr{B}^{-1}(L_2+L_3)L_1^{-1}[L_2+(1-\mathbb{P})L_3]\}\tilde{v}(s) + v(0) + O(\mathscr{B}^{-2}) \tag{A12}$$

where

$$\tilde{v}(s) = \int_0^\infty \exp(-st)v(t)\,dt.$$

Thus in the limit $\mathscr{B} \to \infty$ we have

$$\frac{\partial}{\partial t}\pi(x, t) = L_3\pi(x, t) \tag{A13}$$

the FP equation stochastically equivalent to the SDE (51) within the Itô interpretation. This result allows us to identify the intensity variable z as, in Haken's terminology [35], a silent slave to the scattering power variable x. In this limit we may write the propagator (28) as

$$\begin{aligned} G(x, z, t\,|\,x_0, z_0) &= (1/x)\exp(-z/x)\pi(x, t\,|\,x_0) \qquad t>0 \\ &= \delta(x-x_0)\delta(z-z_0) \qquad t=0 \end{aligned} \tag{A14}$$

where $\pi(x, t\,|\,x_0)$ satisfies (A13) with the initial condition

$$\pi(x, 0\,|\,x_0) = \delta(x-x_0)$$

i.e.

$$\pi(x, t\,|\,x_0) = \frac{1}{(1-e^{-t})}\left(\frac{x\,e^t}{x_0}\right)^{\nu/2}\exp\left(-\frac{1}{(1-e^{-t})}(x+x_0\,e^{-t})\right)I_\nu\left(\frac{2\exp(-t/2)(xx_0)^{1/2}}{1-e^{-t}}\right).$$

Substitution of (A14) into (30) gives the result (44) in which $\mathscr{B}$ has been allowed to approach infinity. To obtain a solution to higher order in $\mathscr{B}^{-1}$ we must generate corrections to (A14) (cf the results of § 6.6.3 in Gardiner's text) and attempt to solve the resulting equations. This, and the equally daunting task of evaluating the memory function in (38), will not be pursued here.

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